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Braids, shuffles and symmetrizers*

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Abstract

Multiplicative analogues of the shuffle elements of the braid group rings are introduced; in local representations they give rise to certain graded associative algebras (b-shuffle algebras). For the Hecke and BMW algebras, the (anti)-symmetrizers have simple expressions in terms of the multiplicative shuffles. The (anti)-symmetrizers can be expressed in terms of the highest multiplicative 1-shuffles (for the Hecke and BMW algebras) and in terms of the highest additive 1-shuffles (for the Hecke algebras). The spectra and multiplicities of eigenvalues of the operators of the multiplication by the multiplicative and additive 1-shuffles are examined.

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1. Braid shuffles

In this section we collect some necessary information on shuffle elements in the braid group rings.

In the Artin presentation, the braid group B_{M+1} is given by generators σ_i, $1 \leqslant i \leqslant M$, and relations

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \qquad \text{if} \quad |i-j| = 1, \tag{1}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if} \quad |i-j| > 1. \tag{2}$$

The inductive limit $B_\infty = \overset{\lim}{\longrightarrow} B_M$ is defined by inclusions $B_M \to B_{M+1}$, $B_M \ni \sigma_i \mapsto \sigma_i \in B_{M+1}$, $i = 1, \dots, M-1$.

We denote $w^{\uparrow \ell}$, as in [28], the image of an element $w \in B_\infty$ under the endomorphism of B_∞, sending σ_i to $\sigma_{i+\ell}$, $i = 1, 2, \dots$ (we keep the same notation for the Hecke and BMW quotients of the braid group rings).

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Braid shuffle elements $Ш_{m,n}$ $(m, n \in \mathbb{Z}_{\geqslant 0})$ are analogues of the binomial coefficients. The shuffle elements belong to the group ring of B_{m+n} (and thereby of B_∞); they can be defined inductively by any of the recurrence relations (braid analogues of the Pascal rule)

$$Ш_{m,n} = Ш_{m-1,n} + Ш_{m,n-1}\sigma_{m+n-1}\cdots\sigma_n, \tag{3}$$

$$Ш_{m,n} = Ш^{\uparrow 1}_{m,n-1} + Ш^{\uparrow 1}_{m-1,n}\sigma_1\cdots\sigma_n, \tag{4}$$

together with the boundary conditions $Ш_{0,n} = 1$ and $Ш_{n,0} = 1$ for any non-negative integer n.

Let Σ_n be the lift [23] of the symmetrizer $\sum_{g\in S_n} g$ from the symmetric group ring $\mathbb{Z}\mathbb{S}_n$ to $\mathbb{Z}B_n$. The element Σ_n is the braid analogue of $n!$; it satisfies

$$\Sigma_{m+n} = Ш_{n,m}\Sigma_m\Sigma^{\uparrow m}_n. \tag{5}$$

Using the automorphism $\mathfrak{a}$ and the anti-automorphism $\mathfrak{b}$, $\mathfrak{b}(xy) = \mathfrak{b}(y)\mathfrak{b}(x)$, of the braid group B_{n+1}, defined on the generators by

$$\mathfrak{a} : \sigma_i \mapsto \sigma_{n+1-i}, \qquad \mathfrak{b} : \sigma_i \mapsto \sigma_i, \tag{6}$$

and their composition, one obtains three more decompositions of Σ.

Higher shuffles (braid analogues of the trinomial, etc coefficients) appear in the further decompositions of the elements Σ_n

$$\Sigma_{m+n+k} = \begin{cases} Ш_{n+k,m}\Sigma_m\Sigma^{\uparrow m}_{n+k} = Ш_{n+k,m}Ш^{\uparrow m}_{k,n}\Sigma_m\Sigma^{\uparrow m}_n\Sigma^{\uparrow m+n}_k, \\ Ш_{k,m+n}\Sigma_{m+n}\Sigma^{\uparrow m+n}_k = Ш_{k,m+n}Ш_{n,m}\Sigma_m\Sigma^{\uparrow m}_n\Sigma^{\uparrow m+n}_k. \end{cases} \tag{7}$$

Due to the existence of a (one-sided) order on the braid groups [5], the braid group rings $\mathbb{Z}B_n$ have no zero divisors. Equating the two expressions for Σ_{m+n+k} in (7) and simplifying, one finds

$$Ш_{n+k,m}Ш^{\uparrow m}_{k,n} = Ш_{k,m+n}Ш_{n,m}. \tag{8}$$

A direct verification of (8) is a good exercise. Any of the expressions in (8) is the braid trinomial coefficient $Ш_{k,n,m}$. The element Σ_n is the shuffle $Ш_{1,1,\ldots,1}$.

We shall later use the following identity:

$$Ш_{1,n-1}Ш_{1,n-2}\cdots Ш_{1,n-k} = Ш_{k,n-k}\Sigma^{\uparrow n-k}_k, \tag{9}$$

which is verified by induction. For $k = 1$ there is nothing to prove. The induction step uses (8) and then (5)

$$\begin{aligned} Ш_{k,n-k}\Sigma^{\uparrow n-k}_k Ш_{1,n-k-1} &= Ш_{k,n-k}Ш_{1,n-k-1}\Sigma^{\uparrow n-k}_k \\ &= Ш_{k+1,n-k-1}Ш^{\uparrow n-k-1}_{k,1}\Sigma^{\uparrow n-k}_k = Ш_{k+1,n-k-1}\Sigma^{\uparrow n-k-1}_{k+1}. \end{aligned} \tag{10}$$

Shuffle elements find numerous applications in the theories of free Lie algebras, polylogarithms and multiple zeta values, Hopf algebras, differential calculus on quantum groups, homology of quantum Lie algebras, braidings of tensor spaces, etc [29, 3, 26, 1, 30, 34, 15, 8, 9].

2. B-shuffle algebras

In this section we recall the definition of the Nichols–Woronowicz algebras and construct, with the help of the baxterized elements, another family of graded associative algebras in the tensor spaces of local representations of the braid groups.

(1) Let V be a vector space over a field $\mathfrak{k}$. For an operator $X \in \mathrm{End}(V^{\otimes j})$ we denote by the same symbol the operator $X \otimes \mathrm{Id}^{\otimes l} \in \mathrm{End}(V^{\otimes(j+l)})$ for any $l \in \mathbb{Z}_{\geqslant 0}$; $X^{\uparrow l}$ denotes the operator $\mathrm{Id}^{\otimes l} \otimes X \in \mathrm{End}(V^{\otimes(j+l)})$.

Let $\{\mathcal{T}_{m,n}\}_{m,n \in \mathbb{Z}_{\geqslant 0}}$ be a collection of operators, $\mathcal{T}_{m,n} \in \mathrm{End}(V^{\otimes(m+n)})$, such that

$$\mathcal{T}_{n+k,m}\mathcal{T}_{k,n}^{\uparrow m} = \mathcal{T}_{k,m+n}\mathcal{T}_{n,m} \qquad \forall\, m,n,k \in \mathbb{Z}_{\geqslant 0}. \tag{11}$$

For tensors $u \in V^{\otimes m}$ and $v \in V^{\otimes n}$ let

$$u \odot v := \mathcal{T}_{n,m}(u \otimes v) \in V^{\otimes(m+n)}. \tag{12}$$

Due to (11), the space $\bigoplus_j V^{\otimes j}$ with the composition law $\odot$ is an associative graded algebra. Assume, in addition, that

$$\mathcal{T}_{m,0} = \mathrm{Id} \qquad \text{and} \qquad \mathcal{T}_{0,m} = \mathrm{Id} \qquad \forall\, m \in \mathbb{Z}_{\geqslant 0} \tag{13}$$

(then $1 \in \mathfrak{k} \equiv V^{\otimes 0}$ is the identity element of the algebra). By (11) and (13), the following collection $\{\mathcal{S}_m\}_{m \in \mathbb{Z}_{\geqslant 0}}$ of operators, $\mathcal{S}_m \in \mathrm{End}(V^{\otimes m})$:

$$\mathcal{S}_0 = \mathrm{Id}, \qquad \mathcal{S}_1 = \mathrm{Id}, \qquad \mathcal{S}_{m+n} = \mathcal{T}_{n,m}\mathcal{S}_m\mathcal{S}_n^{\uparrow m} \qquad \forall\, m,n \in \mathbb{Z}_{\geqslant 0}, \tag{14}$$

is well defined. The operation $\odot$ restricts on $\bigoplus_j \mathrm{Im}(\mathcal{S}_j)$, the direct sum of images of the operators $\mathcal{S}_j$, making it an associative graded algebra as well.

Let $\hat{R} \in \mathrm{End}(V \otimes V)$ be a solution of the Yang–Baxter equation, that is, $\hat{R}\hat{R}^{\uparrow 1}\hat{R} = \hat{R}^{\uparrow 1}\hat{R}\hat{R}^{\uparrow 1}$. Denote by $\rho_{\hat{R}}$ the corresponding local representation of the braid groups B_n, $\rho_{\hat{R}}(\sigma_i) := \hat{R}^{\uparrow(i-1)}$. Then the collection $\mathcal{T}_{m,n} := \rho_{\hat{R}}(\text{Ш}_{m,n})$ obeys (11) and (13). The space $\bigoplus_j \mathrm{Im}\,\rho_{\hat{R}}(\Sigma_j)$ with the composition law $\odot$ is called the Nichols–Woronowicz algebra.

(2) The braid group rings admit a family of automorphisms $\sigma_i \mapsto t\sigma_i$, where $t \in \mathfrak{k}^*$ is an arbitrary parameter. The formal limits $\lim_{t\to 0}$ (the lowest power in t) and $\lim_{t\to\infty}$ (the highest power in t) of the elements Σ_m, $\text{Ш}_{m,n}$ and the operation $\odot$ are well defined. For $t \to 0$ we obtain the usual tensor algebra, while for $t \to \infty$ the element Σ_{n+1} becomes the lift of the longest element of the symmetric group $\mathbb{S}_{n+1}$ to B_{n+1}

$$\bar{\Sigma}_{n+1} = (\sigma_1\sigma_2\cdots\sigma_n)(\sigma_1\cdots\sigma_{n-1})\cdots(\sigma_1). \tag{15}$$

The shuffle elements, in the limit $\lim_{t\to\infty}$, become the elements $\bar{\text{Ш}}_{m,n}$ which, in a representation in a vector space V, equip the tensor powers of V with the standard braidings; the recurrency relations (3) and (4) take the multiplicative form for $\bar{\text{Ш}}_{m,n}$

$$\bar{\text{Ш}}_{m,n} = \bar{\text{Ш}}_{m,n-1}\sigma_{m+n-1}\cdots\sigma_n, \qquad \bar{\text{Ш}}_{m,n} = \bar{\text{Ш}}_{m-1,n}^{\uparrow 1}\sigma_1\cdots\sigma_n. \tag{16}$$

Explicitly

$$\bar{\text{Ш}}_{m,n} = \begin{cases} (\sigma_m\sigma_{m+1}\cdots\sigma_{m+n-1})(\sigma_{m-1}\sigma_m\cdots\sigma_{m+n-2})\cdots(\sigma_1\sigma_2\cdots\sigma_n), \\ (\sigma_m\sigma_{m-1}\cdots\sigma_1)(\sigma_{m+1}\sigma_m\cdots\sigma_2)\cdots(\sigma_{m+n-1}\sigma_{m+n-2}\cdots\sigma_n). \end{cases} \tag{17}$$

In addition to (16), the elements $\bar{\text{Ш}}_{m,n}$ satisfy

$$\bar{\text{Ш}}_{m,n} = \sigma_m\cdots\sigma_1\bar{\text{Ш}}_{m,n-1}^{\uparrow 1}, \qquad \bar{\text{Ш}}_{m,n} = \sigma_m\cdots\sigma_{m+n-1}\bar{\text{Ш}}_{m-1,n}. \tag{18}$$

(3) In this section we shall construct another collection $\mathcal{T}_{m,n}$ starting with the elements $\sigma_k(x,y)$, satisfying the Yang–Baxter equation with spectral parameters

$$\sigma_k(x_{k+1},x_{k+2})\sigma_{k+1}(x_k,x_{k+2})\sigma_k(x_k,x_{k+1}) = \sigma_{k+1}(x_k,x_{k+1})\sigma_k(x_k,x_{k+2})\sigma_{k+1}(x_{k+1},x_{k+2}) \tag{19}$$

and the locality condition

$$\sigma_k(x_k,x_{k+1})\sigma_l(x_l,x_{l+1}) = \sigma_l(x_l,x_{l+1})\sigma_k(x_k,x_{k+1}) \qquad \text{if} \quad |k-l| > 1. \tag{20}$$

Here x_k are variables (spectral parameters). Depending on the situation, the elements $\sigma_k(x, y)$ can live in certain quotients of the braid group rings or be realized as operators. We shall call $\sigma_k(x, y)$ baxterized elements (usually the term 'baxterized' is applied when $\sigma(x, y)$ is a function of the solution σ of the constant Yang–Baxter equation).

Let π_k be the operator which permutes the variables x_k and x_{k+1}

$$\pi_k f(\ldots, x_k, x_{k+1}, \ldots) = f(\ldots, x_{k+1}, x_k, \ldots)\pi_k.$$

Relations (19) and (20) acquire the braid forms (1) and (2) for the elements

$$\underline{\sigma}_k := \pi_k \sigma_k(x_k, x_{k+1}). \tag{21}$$

The unitarity condition $\sigma_k(x_k, x_{k+1})\sigma_k(x_{k+1}, x_k) = 1$ (if imposed) for the baxterized elements takes the form $\underline{\sigma}_k^2 = 1$ for the elements (21).

The operators π_k obey the braid group relations; prepare the elements $\bar{\text{III}}_{m,n}\{\pi\}$ and $\bar{\Sigma}_m\{\pi\}$ from π's; the elements $\bar{\text{III}}_{m,n}\{\underline{\sigma}\}$ and $\bar{\Sigma}_m\{\underline{\sigma}\}$ built from $\underline{\sigma}$'s can be written, after moving all π's to the left, in the form

$$\bar{\text{III}}_{m,n}\{\underline{\sigma}\} = \bar{\text{III}}_{m,n}\{\pi\}\widetilde{\text{III}}_{m,n}(x_1, \ldots, x_{m+n}), \qquad \bar{\Sigma}_m\{\underline{\sigma}\} = \bar{\Sigma}_m\{\pi\}\widetilde{\Sigma}_m(x_1, \ldots, x_m), \tag{22}$$

where

$$\begin{aligned}\widetilde{\text{III}}_{m,n}(x_1, \ldots, x_{m+n}) = {}& (\sigma_m(x_1, x_{m+n})\sigma_{m+1}(x_2, x_{m+n}) \cdots \sigma_{m+n-1}(x_n, x_{m+n}))\\ &\cdot (\sigma_{m-1}(x_1, x_{m+n-1})\sigma_m(x_2, x_{m+n-1}) \cdots \sigma_{m+n-2}(x_n, x_{m+n-1}))\\ &\cdots (\sigma_1(x_1, x_{n+1})\sigma_2(x_2, x_{n+1}) \cdots \sigma_n(x_n, x_{n+1}))\end{aligned} \tag{23}$$

and

$$\begin{aligned}\widetilde{\Sigma}_m(x_1, \ldots, x_m) = {}& (\sigma_1(x_{m-1}, x_m)\sigma_2(x_{m-2}, x_m) \cdots \sigma_{m-1}(x_1, x_m))\\ &\cdot (\sigma_1(x_{m-2}, x_{m-1})\sigma_2(x_{m-3}, x_{m-1}) \cdots \sigma_{m-2}(x_1, x_{m-1})) \cdots (\sigma_1(x_1, x_2)).\end{aligned} \tag{24}$$

The elements $\bar{\text{III}}_{m,n}\{\pi\}$ and $\bar{\Sigma}_n\{\pi\}$ are invertible and obey relations (5), (8), (16) and (18); substituting (22) into (5), (8), (16) and (18), moving all π's to the left and simplifying, we find relations for $\widetilde{\Sigma}$'s and $\widetilde{\text{III}}$'s alone. Relations (16) and (18) take the form

$$\widetilde{\text{III}}_{m,n}(x_1, \ldots, x_{m+n}) = \begin{cases} \widetilde{\text{III}}_{m,n-1}(\hat{x}_n)\sigma_{m+n-1}(x_n, x_{m+n})\sigma_{m+n-2}(x_n, x_{m+n-1}) \cdots \sigma_n(x_n, x_{n+1}),\\ \widetilde{\text{III}}^{\uparrow 1}_{m-1,n}(\hat{x}_{n+1})\sigma_1(x_1, x_{n+1})\sigma_2(x_2, x_{n+1}) \cdots \sigma_n(x_n, x_{n+1}),\\ \sigma_m(x_1, x_{m+n})\sigma_{m-1}(x_1, x_{m+n-1}) \cdots \sigma_1(x_1, x_{n+1})\widetilde{\text{III}}^{\uparrow 1}_{m,n-1}(\hat{x}_1),\\ \sigma_m(x_1, x_{m+n})\sigma_{m+1}(x_2, x_{m+n}) \cdots \sigma_{m+n-1}(x_n, x_{m+n})\widetilde{\text{III}}_{m-1,n}(\hat{x}_{m+n}),\end{cases} \tag{25}$$

where '$\hat{x}_j$' means that the argument x_j is omitted. For a set $\overrightarrow{x} = \{x_1, \ldots, x_n\}$ of arguments, let $\overleftarrow{x} := \{x_n, \ldots, x_1\}$ be the reversed set. Relation (5) becomes

$$\widetilde{\Sigma}_{m+n}(\overrightarrow{x}, \overrightarrow{y}) = \widetilde{\text{III}}_{n,m}(\overleftarrow{x}, \overleftarrow{y})\widetilde{\Sigma}_m(\overrightarrow{x})\widetilde{\Sigma}^{\uparrow m}_n(\overrightarrow{y}), \tag{26}$$

where $\overrightarrow{x} = \{x_1, \ldots, x_m\}$ and $\overrightarrow{y} = \{y_1, \ldots, y_n\}$; relation (8) becomes

$$\widetilde{\text{III}}_{n+k,m}(\overrightarrow{x}, \overrightarrow{z}, \overrightarrow{y})\widetilde{\text{III}}^{\uparrow m}_{k,n}(\overrightarrow{y}, \overrightarrow{z}) = \widetilde{\text{III}}_{k,m+n}(\overrightarrow{y}, \overrightarrow{x}, \overrightarrow{z})\widetilde{\text{III}}_{n,m}(\overrightarrow{x}, \overrightarrow{y}), \tag{27}$$

where $\overrightarrow{x} = \{x_1, \ldots, x_m\}$, $\overrightarrow{y} = \{y_1, \ldots, y_n\}$ and $\overrightarrow{z} = \{z_1, \ldots, z_k\}$.

After the removal of all π's, one can give values to the spectral variables. Each $\widetilde{\Sigma}_m$ can be evaluated on its own sequence $\vec{x}^{(m)} = \left(x_1^{(m)}, \ldots, x_m^{(m)}\right)$. In relation (26), the beginning of the sequence for $\widetilde{\Sigma}_{m+n}$ becomes the beginning of the sequence for $\widetilde{\Sigma}_m$ while its end becomes the beginning of the sequence for $\widetilde{\Sigma}_n$. This is a strong restriction; if it is imposed on the sequences themselves, the general solution is that each $x_j^{(m)}$ is equal to one and the same number. However, assume that the baxterization is 'trigonometric', the baxterized elements depend on the ratio of the spectral parameters, $\sigma(x, y) = \sigma(x/y)$. The Yang–Baxter equation then reads

$$\sigma_n(x)\sigma_{n-1}(xy)\sigma_n(y) = \sigma_{n-1}(y)\sigma_n(xy)\sigma_{n-1}(x). \tag{28}$$

Now $\widetilde{\Sigma}_m(\vec{x}^{(m)}) = \widetilde{\Sigma}_m(\alpha\vec{x}^{(m)})$ for an arbitrary constant $\alpha \neq 0$ and the general solution of the restrictions imposed by (26) for the projectivized sequences is $\left(x_1^{(m)}, \ldots, x_m^{(m)}\right) = (1, s^{-1}, s^{-2}, \ldots, s^{1-m})$, the geometric progression. Denote $\widetilde{\Sigma}_m(1, s^{-1}, s^{-2}, \ldots, s^{1-m})$ by ${}^s\Sigma_m$ and $\widetilde{\text{Ш}}_{m,n}(s^{1-n}, \ldots, 1, s^{1-m-n}, \ldots, s^{-n})$ by ${}^s\text{Ш}_{m,n}$. Explicitly

$${}^s\Sigma_m = (\sigma_1(s)\sigma_2(s^2)\cdots\sigma_{m-1}(s^{m-1}))(\sigma_1(s)\sigma_2(s^2)\cdots\sigma_{m-2}(s^{m-2}))\cdots(\sigma_1(s)) \tag{29}$$

and

$${}^s\text{Ш}_{m,n} = (\sigma_m(s)\cdots\sigma_{m+n-1}(s^n))(\sigma_{m-1}(s^2)\cdots\sigma_{m+n-2}(s^{n+1}))\cdots(\sigma_1(s^m)\cdots\sigma_n(s^{m+n-1})). \tag{30}$$

The elements ${}^s\text{Ш}_{m,n}$ obey relation (8). Therefore, in a local representation ρ, the collection $\mathcal{T}_{m,n} := \rho_{\hat{R}}({}^s\text{Ш}_{m,n})$ obeys (11) and (13) and defines a one-parameter family of graded associative algebras on $\bigoplus_j V^{\otimes j}$ together with the subalgebras on $\bigoplus_j \text{Im}(\mathcal{S}_j)$ (now $\mathcal{S}_m = \rho_{\hat{R}}({}^s\Sigma_m)$), which we propose to call *b-shuffle algebras* ('b' from 'baxterized'; maybe the term 'buffle' would be an apt acronym).

It is known that the element $\bar{\Sigma}$ admits reduced expressions starting (or ending) with σ_j for every $j = 1, \ldots, m-1$. In particular, $\bar{\Sigma}\{\underline{\sigma}\}$ can start (or end) with every $\underline{\sigma}_j$. It follows that $\widetilde{\Sigma}_m(x_1, \ldots, x_m)$ can start (or end) with $\sigma_j(x_j, x_{j+1})$ for every j. We shall use this for the trigonometric σ's.

$${}^s\Sigma_m \text{ has a reduced expression of the form } \sigma_j(s)\cdot(\cdots) \text{ or } (\cdots)\cdot\sigma_j(s)\forall j = 1, \ldots, m-1. \tag{31}$$

The baxterization is known for the Hecke and BMW quotients of the braid group rings; it is trigonometric. In the following section we discuss the baxterized collections for these quotients.

Remarks.

(a) We suggest another natural source for collections $\mathcal{T}_{m,n}$ satisfying (11) and (13).

Let $\mathcal{A}$ be a Hopf algebra. Assume that $\mathcal{A}$ admits a twist $\mathcal{F}$, that is, an element $\mathcal{F} \in \mathcal{A}\otimes\mathcal{A}$ which satisfies

$$\mathcal{F}\cdot(\Delta\otimes\text{Id})(\mathcal{F}) = \mathcal{F}^{\uparrow 1}\cdot(\text{Id}\otimes\Delta)(\mathcal{F}). \tag{32}$$

Here Δ is the coproduct and $^{\uparrow}$ is the shift in the copies of $\mathcal{A}$ in $\mathcal{A}^{\otimes j}$. Define $\mathcal{F}_{m,0} := 1$, $\mathcal{F}_{0,m} := 1$, $m \in \mathbb{Z}_{\geqslant 0}$, and

$$\mathcal{F}_{m,n} := \Delta^{m-1}\otimes\Delta^{n-1}(\mathcal{F}), \qquad m, n \in \mathbb{Z}_{\geqslant 1}. \tag{33}$$

It is straightforward to verify that

$$\mathcal{F}_{k,m}\mathcal{F}_{m+k,n} = \mathcal{F}_{m,n}^{\uparrow k}\mathcal{F}_{k,m+n} \tag{34}$$

(for $m = n = k = 1$ this is (32); by induction, $\text{Id}^{i-1}\otimes\Delta\otimes\text{Id}^{m+n+k-i}$ increases k by 1 for $1 \leqslant i \leqslant k$, m by 1 for $k < i \leqslant m+k$ and n by 1 for $m+k < i \leqslant m+n+k$).

Therefore, given a representation ρ of $\mathcal{A}$, relations (11) and (13) hold for

$$\mathcal{T}_{m,n} := \varpi \circ \rho^{\otimes(m+n)}(\mathcal{F}_{n,m}),$$

where ϖ is any operation which reverses the order of terms on both sides of (34) (it can be a transposition or, if $\mathcal{F}_{m,n}$ are invertible for all m and n, an inversion).

It might be of interest to investigate this type of collections $\mathcal{T}_{m,n}$ for the twists [14] corresponding to Belavin–Drinfeld triples.

(b) Assume, in addition, that $\mathcal{F}$ satisfies

$$(\Delta \otimes \mathrm{Id})(\mathcal{F}) = \mathcal{F}_{\{1,3\}}\mathcal{F}_{\{2,3\}}, \qquad (\mathrm{Id} \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{\{1,3\}}\mathcal{F}_{\{1,2\}}, \tag{35}$$

where $\mathcal{F}_{\{i,j\}}$ is the element $\mathcal{F}$ located in the copies number i and j in $\mathcal{A} \otimes \mathcal{A} \otimes \cdots$; for example, for a quasi-triangular Hopf algebra, $\mathcal{F}$ can be the universal R-matrix. Then

$$\mathcal{F}_{m,n} = (\mathcal{F}_{\{1,m+n\}} \cdots \mathcal{F}_{\{m,m+n\}})(\mathcal{F}_{\{1,m+n-1\}} \cdots \mathcal{F}_{\{m,m+n-1\}}) \cdots (\mathcal{F}_{\{1,m+1\}} \cdots \mathcal{F}_{\{m,m+1\}}) \tag{36}$$

(in each bracket the first index increases from 1 to m, the second one is constant); this formula generalizes the formula $\Delta \otimes \Delta(\mathcal{R}) = \mathcal{R}_{\{1,4\}}\mathcal{R}_{\{2,4\}}\mathcal{R}_{\{1,3\}}\mathcal{R}_{\{2,3\}}$ used in the theory of quasi-triangular Hopf algebras for establishing properties of the element giving the square of the antipode by conjugation, see, e.g., [[27], chapter 4]. It follows from (36) that:

$$\mathcal{F}_{m,n} = (\mathcal{F}_{\{1,m+n\}}\mathcal{F}_{\{1,m+n-1\}} \cdots \mathcal{F}_{\{1,m+1\}})\mathcal{F}^{\uparrow 1}_{m-1,n}. \tag{37}$$

Given a representation ρ of $\mathcal{A}$ on a vector space V, let P_i be the flip operator in the copies number i and $i+1$ of the space V in $V \otimes V \otimes \cdots$; let $F := \rho^{\otimes 2}(\mathcal{F})$ and $\hat{F} := P_1 F$; for an operator $X \in \mathrm{End}(V \otimes V)$ denote by $X_{\{i,j\}}$ the operator X acting in the copies number i and j of the space V in $V \otimes V \otimes \cdots$ and let $X_i := X_{\{i,i+1\}}$. Then

$$\rho^{\otimes(m+n)}(\mathcal{F}_{m,n}) = \bar{\mathrm{III}}_{m,n}\{P\}\bar{\mathrm{III}}_{n,m}\{\hat{F}\}, \tag{38}$$

where $\bar{\mathrm{III}}_{m,n}\{P\}$ are built from P's and $\bar{\mathrm{III}}_{n,m}\{\hat{F}\}$ from $\hat{F}$'s. Indeed, by (37) and induction

$$\begin{aligned}\rho^{\otimes(m+n)}(\mathcal{F}_{m,n}) &= (F_{\{1,m+n\}} \cdots F_{\{1,m+1\}})\bar{\mathrm{III}}^{\uparrow 1}_{m-1,n}\{P\}\bar{\mathrm{III}}^{\uparrow 1}_{n,m-1}\{\hat{F}\} \\ &= (P_{\{1,m+n\}} \cdots P_{\{1,m+1\}})(\hat{F}_{m+n-1}\hat{F}_{m+n-2} \cdots \hat{F}_{m+1}\hat{F}_{\{1,m+1\}}\bar{\mathrm{III}}^{\uparrow 1}_{m-1,n}\{P\}\bar{\mathrm{III}}^{\uparrow 1}_{n,m-1}\{\hat{F}\}.\end{aligned} \tag{39}$$

Use now

$$(\hat{F}_{m+n-1}\hat{F}_{m+n-2} \cdots \hat{F}_{m+1}\hat{F}_{\{1,m+1\}})\bar{\mathrm{III}}^{\uparrow 1}_{m-1,n}\{P\} = \bar{\mathrm{III}}^{\uparrow 1}_{m-1,n}\{P\}(\hat{F}_n\hat{F}_{n-1} \cdots \hat{F}_1), \tag{40}$$

the first recursion relation in (18) for $\bar{\mathrm{III}}_{n,m}\{\hat{F}\}$ and

$$\begin{aligned}(P_{\{1,m+n\}} \cdots P_{\{1,m+1\}})\bar{\mathrm{III}}^{\uparrow 1}_{m-1,n}\{P\} &= \bar{\mathrm{III}}^{\uparrow 1}_{m-1,n}\{P\}(P_{\{1,n+1\}}P_{\{1,n\}} \cdots P_{\{1,2\}}) \\ &= \bar{\mathrm{III}}^{\uparrow 1}_{m-1,n}\{P\}(P_1P_2 \cdots P_n) = \bar{\mathrm{III}}_{m,n}\{P\}\end{aligned} \tag{41}$$

(by the second recurrency relation in (16) for $\bar{\mathrm{III}}_{m,n}\{P\}$) to finish the proof of (38).

Thus the elements $\mathcal{F}_{m,n}$ can be regarded as the universal (in the Hopf algebra theoretical sense) counterpart of the elements $\bar{\mathrm{III}}_{m,n}$.

(c) We describe an operation which transforms a collection $\mathcal{T}_{m,n}$ satisfying (11) and (13) into another, 'dual', collection $\check{\mathcal{T}}_{m,n}$ satisfying (11) and (13).

Keep the notation from the previous remark. Let $X \in \mathrm{End}(V^{\otimes m})$ and $Y \in \mathrm{End}(V^{\otimes n})$ be two operators. Then

$$\bar{\mathrm{III}}_{m,n}\{P\}X^{\uparrow n}Y = XY^{\uparrow m}\bar{\mathrm{III}}_{m,n}\{P\}. \tag{42}$$

Define $\check{\mathcal{T}}_{m,n}$ by

$$\mathcal{T}_{m,n} := \check{\mathcal{T}}_{n,m} \bar{Ш}_{m,n}\{P\} \qquad \text{or} \qquad \check{\mathcal{T}}_{m,n} := \mathcal{T}_{n,m} \bar{Ш}_{m,n}\{P\}. \tag{43}$$

The equivalence of two definitions follows from:

$$\bar{Ш}_{m,n}\{P\}^{-1} = \bar{Ш}_{n,m}\{P\}. \tag{44}$$

Relation (13) is satisfied for the collection $\check{\mathcal{T}}_{m,n}$. Relation (11) reads, by (42),

$$\check{\mathcal{T}}_{m,n+k} \check{\mathcal{T}}_{n,k} \bar{Ш}_{n+k,m}\{P\} \bar{Ш}^{\uparrow m}_{k,n}\{P\} = \check{\mathcal{T}}_{m+n,k} \check{\mathcal{T}}^{\uparrow k}_{m,n} \bar{Ш}_{k,m+n}\{P\} \bar{Ш}_{n,m}\{P\}. \tag{45}$$

Since

$$\bar{Ш}_{n+k,m}\{P\} \bar{Ш}^{\uparrow m}_{k,n}\{P\} = \bar{Ш}_{k,m+n}\{P\} \bar{Ш}_{n,m}\{P\} \tag{46}$$

it follows that relation (11) is as well satisfied for the collection $\check{\mathcal{T}}_{m,n}$.

With the help of the identity $\bar{\Sigma}_m\{P\}^2 = \mathrm{Id}$, it is straightforward to verify that the collection $\check{\mathcal{S}}_m$ for $\check{\mathcal{T}}_{m,n}$ is given by

$$\check{\mathcal{S}}_m = \mathcal{S}_m \bar{\Sigma}_m\{P\}. \tag{47}$$

3. Hecke and BMW algebras

In the following we call the elements $Ш_{m,n}$ additive shuffles and ${}^s Ш_{m,n}$ multiplicative shuffles. In this section we derive the sequences of the (anti-)symmetrizers for the Hecke and BMW algebras with the help of the multiplicative shuffles. We compare the multiplicative versions with known expressions for the (anti-)symmetrizers.

We derive a new expression for the (anti-)symmetrizers in terms of the highest multiplicative 1-shuffles alone and, for the Hecke algebras, in terms of the highest additive 1-shuffles alone.

In principle, the Hecke algebras can be considered as quotients of the BMW algebras and many formulae for the Hecke algebras can be obtained from this point of view. Because of the importance of the Hecke algebras we prefer however to treat them separately.

3.1. Hecke algebras

(1) The tower of the A-Type Hecke algebras $H_{M+1}(q)$ (see, e.g., [19] and references therein) depends on a parameter $q \in \mathfrak{k}^*$; the algebra $H_{M+1}(q)$ is the quotient of the braid group ring $\mathfrak{k}B_{M+1}$ by

$$\sigma_i^2 = (q - q^{-1})\sigma_i + 1, \qquad i = 1, \ldots, M. \tag{48}$$

For $q^2 \neq 1$, the baxterized elements have the form

$$\sigma_i(x) := \frac{1}{q - q^{-1}} \left(x\sigma_i - x^{-1}\sigma_i^{-1} \right); \tag{49}$$

they are normalized, $\sigma_i(1) = 1$, and satisfy the unitarity condition

$$\sigma_i(x)\sigma_i(x^{-1}) = 1 - \frac{(x - x^{-1})^2}{(q - q^{-1})^2}. \tag{50}$$

(2) The symmetrizers $S_n, n = 1, \ldots, M+1$ ([18, 32, 10]) are the nonzero elements

$$S_1 = 1, \qquad S_n \in H_n(q) \subset H_{M+1}(q), \tag{51}$$

which satisfy

$$\sigma_i S_n = S_n \sigma_i = q S_n, \qquad i = 1, \ldots, n-1, \tag{52}$$

which forces $S_n^2 \sim S_n$; they are normalized by

$$S_n^2 = S_n. \tag{53}$$

The sequence $\{S_n\}$ is defined by (51), (52) and (53) uniquely (and it does exist for the Hecke quotients for generic q); the anti-symmetrizers are related to the symmetrizers by the isomorphisms $H_{M+1}(q) \to H_{M+1}(-q^{-1}), \sigma_i \mapsto \sigma_i$.

(3) The symmetrizers can be quickly constructed with the help of the baxterized elements. Let $[n]_q := (q^n - q^{-n})/(q - q^{-1})$, $[n]_q! := [1]_q[2]_q \cdots [n]_q$ and $[n]_q^{\$} := [1]_q![2]_q! \cdots [n]_q!$. By (31), $\sigma_i {}^q\Sigma_n = {}^q\Sigma_n \sigma_i = q\,{}^q\Sigma_n$, or, equivalently, $\sigma_i(q)\,{}^q\Sigma_n = [i+1]_q\,{}^q\Sigma_n$, so $({}^q\Sigma_n)^2 = [n]_q^{\$}\,{}^q\Sigma_n$ and

$$S_n = \frac{1}{[n]_q^{\$}}\,{}^q\Sigma_n. \tag{54}$$

In particular, the symmetrizers satisfy the recurrent relation

$$S_n = \frac{1}{[n]_q!}\,{}^q\text{Ш}_{1,n-1} S_{n-1}. \tag{55}$$

(4) We recall several other forms of the symmetrizers and compare them with (54) and (55). A convenient recurrent relation for the symmetrizers is (see, e.g., [17, 11])

$$S_n = S_{n-1}\frac{\sigma_{n-1}(q^{n-1})}{[n]_q} S_{n-1}. \tag{56}$$

This is checked either by verifying (51), (52) and (53) and then by uniqueness or, using (55), by the following calculation:

$$\begin{aligned}[n]_q! S_n &= \sigma_1(q)\cdots\sigma_{n-2}(q^{n-2})\sigma_{n-1}(q^{n-1})S_{n-1}\\ &= \sigma_1(q)\cdots\sigma_{n-2}(q^{n-2})\sigma_{n-1}(q^{n-1})S_{n-2}S_{n-1}\\ &= \sigma_1(q)\cdots\sigma_{n-2}(q^{n-2})S_{n-2}\sigma_{n-1}(q^{n-1})S_{n-1} = S_{n-1}\sigma_{n-1}(q^{n-1})S_{n-1}.\end{aligned} \tag{57}$$

Denote $\text{Ш}_{1,n}\{q\sigma\}$ (the additive shuffle built with the $q\sigma_1, \ldots, q\sigma_{n-1}$) by $\mathbf{Ш}_{1,n}$. There is another recurrent relation for the symmetrizers in terms of the additive shuffles

$$S_n = \frac{q^{1-n}}{[n]_q}\mathbf{Ш}_{1,n-1} S_{n-1}. \tag{58}$$

In other words

$$S_n = \frac{q^{-\frac{n(n-1)}{2}}}{[n]_q!}\Sigma_n\{q\sigma\}. \tag{59}$$

This is checked again either by verifying (51), (52) and (53) and then by uniqueness or, using (56), by induction

$$\begin{aligned}[n]_q S_n &= S_{n-1}\sigma_{n-1}(q^{n-1})S_{n-1} = \frac{q^{2-n}}{[n-1]_q}\mathbf{Ш}_{1,n-2}S_{n-2}\sigma_{n-1}(q^{n-1})S_{n-1}\\ &= \frac{q^{2-n}}{[n-1]_q}\mathbf{Ш}_{1,n-2}\sigma_{n-1}(q^{n-1})S_{n-1} = \frac{q^{2-n}}{[n-1]_q}\mathbf{Ш}_{1,n-2}([n-1]_q\sigma_{n-1} + q^{1-n})S_{n-1}\\ &= \frac{q^{2-n}}{[n-1]_q}([n-1]_q\mathbf{Ш}_{1,n-2}\sigma_{n-1} + q^{-1}[n-1]_q)S_{n-1} = q^{1-n}\mathbf{Ш}_{1,n-1}S_{n-1}.\end{aligned} \tag{60}$$

We stress that the factors $\frac{1}{[n]_q!}{}^{q}\text{Ш}_{1,n-1}$ in (55) and $\frac{q^{1-n}}{[n]_q}\mathbf{Ш}_{1,n-1}$ in (58) differ; the multiplicative and additive shuffles do not coincide although their products—the symmetrizers—do.

(5) It turns out that the symmetrizer S_n can be expressed in terms of the multiplicative 1-shuffle ${}^{q}\text{Ш}_{1,n-1}$ or in terms of the additive 1-shuffle $\mathbf{Ш}_{1,n-1}$ only.

For the additive shuffle, we prove by induction that, for $k = 1, \ldots, n-1$,

$$\prod_{j=0}^{k-1}(\mathbf{Ш}_{1,n-1} - q^{j-1}[j]_q) = q^{k(k-1)}\mathbf{Ш}_{1,n-1}\mathbf{Ш}_{1,n-2}\cdots\mathbf{Ш}_{1,n-k}. \tag{61}$$

For $k = 1$ there is nothing to prove. Assume that (61) holds for some $k < n-1$. By (9), the right-hand side is divisible, from the right, by $S_k^{\uparrow(n-k)}$. Multiply (61) by the factor $(\mathbf{Ш}_{1,n-1} - q^{k-1}[k]_q)$ from the right and substitute, in the right-hand side,

$$\mathbf{Ш}_{1,n-1} = \mathbf{Ш}_{1,k-1}^{\uparrow(n-k)} + q^k\mathbf{Ш}_{1,n-1-k}\sigma_{n-k}\cdots\sigma_{n-1}$$

in this factor. Since $S_k^{\uparrow(n-k)}\mathbf{Ш}_{1,k-1}^{\uparrow(n-k)} = q^{k-1}[k]_q S_k^{\uparrow(n-k)}$, the product in the right-hand side simplifies

$$\begin{aligned}\mathbf{Ш}_{1,n-1}\mathbf{Ш}_{1,n-2}\cdots\mathbf{Ш}_{1,n-k}&\big(-q^{k-1}[k]_q + \mathbf{Ш}_{1,k-1}^{\uparrow(n-k)} + q^k\mathbf{Ш}_{1,n-1-k}\sigma_{n-k}\cdots\sigma_{n-1}\big)\\ &= q^k\mathbf{Ш}_{1,n-1}\mathbf{Ш}_{1,n-2}\cdots\mathbf{Ш}_{1,n-k}\mathbf{Ш}_{1,n-1-k}\sigma_{n-k}\cdots\sigma_{n-1}\\ &= q^{2k}\mathbf{Ш}_{1,n-1}\mathbf{Ш}_{1,n-2}\cdots\mathbf{Ш}_{1,n-1-k}\end{aligned} \tag{62}$$

(in the last equality we again used (9) for $\mathbf{Ш}_{1,n-1}\mathbf{Ш}_{1,n-2}\cdots\mathbf{Ш}_{1,n-k}\mathbf{Ш}_{1,n-1-k}$), establishing the induction step.

In particular, at $k = n-1$, we obtain, by (58), the expression of S_n in terms of $\mathbf{Ш}_{1,n-1}$

$$S_n = \frac{q^{-\frac{(n-1)(3n-4)}{2}}}{[n]_q!}\prod_{j=0}^{n-2}(\mathbf{Ш}_{1,n-1} - q^{j-1}[j]_q). \tag{63}$$

(6) For the multiplicative shuffle, we prove by induction that, for $k = 1, \ldots, n$,

$$({}^{q}\text{Ш}_{1,n})^k = \frac{[n+1-k]_q^{\$}([n+1]_q!)^k}{[n]_q^{\$}}\,{}^{q}\text{Ш}_{1,n}\,{}^{q}\text{Ш}_{1,n-1}\cdots{}^{q}\text{Ш}_{1,n+1-k}. \tag{64}$$

For $k = 1$ there is nothing to prove. Assume that (61) holds for some $k < n$. Relations (5) and (8) hold for $\Sigma_m = {}^{q}\Sigma_m$ and $\text{Ш}_{m,n} = {}^{q}\text{Ш}_{m,n}$. Therefore, (9) holds as well and the product ${}^{q}\text{Ш}_{1,n}{}^{q}\text{Ш}_{1,n-1}\cdots{}^{q}\text{Ш}_{1,n-k}$ is divisible, from the right, by ${}^{q}\Sigma_{k+1}^{\uparrow(n-k)}$. The induction step is

$$\begin{aligned}{}^{q}\text{Ш}_{1,n}\cdots{}^{q}\text{Ш}_{1,n+1-k}\cdot{}^{q}\text{Ш}_{1,n} &= {}^{q}\text{Ш}_{1,n}\cdots{}^{q}\text{Ш}_{1,n+1-k}{}^{q}\text{Ш}_{1,n-k}\cdot\sigma_{n-k+1}(q^{n-k+1})\cdots\sigma_n(q^n)\\ &= \frac{[n+1]_q!}{[n+1-k]_q!}{}^{q}\text{Ш}_{1,n}{}^{q}\text{Ш}_{1,n-1}\cdots{}^{q}\text{Ш}_{1,n+1-k}{}^{q}\text{Ш}_{1,n-k},\end{aligned} \tag{65}$$

since $S_{n+1}\sigma_k(q^k) = [k+1]_q S_{n+1}$, $k = 1, \ldots, n$.

In particular, at $k = n$, we obtain, by (54), the expression of S_n in terms of ${}^q\text{III}_{1,n}$

$$S_{n+1} = \left(\frac{1}{[n+1]_q!} {}^q\text{III}_{1,n} \right)^n . \tag{66}$$

Remark. Let $\hat{R}$ be a Hecke Yang–Baxter matrix and $\rho_{\hat{R}}$ the corresponding local representation of the tower of the Hecke algebras. The symmetrizers $\mathcal{S}_j$ built with $\mathcal{T}'_{m,n} = \rho_{\hat{R}}({}^s\text{III}_{m,n})$, at $s = q$, are the same as the symmetrizers built with $\mathcal{T}''_{m,n} = \rho_{\hat{R}}(\text{III}_{m,n}\{t\sigma\})$, at $t = q$ (the symmetrizers coincide at $s^2 = q^2$ and $t = q$ or $s^2 = q^{-2}$ and $t = -q^{-1}$, these are the values for the anti-symmetrizers; the symmetrizers coincide trivially at $s^2 = 1$ and $t = 0$; otherwise the symmetrizers for $\{\mathcal{T}'_{m,n}\}$ and $\{\mathcal{T}''_{m,n}\}$ differ). Therefore, for the Hecke algebras, the b-shuffle algebra on $\bigoplus_j \text{Im}(\mathcal{S}_j)$ coincides with the Nichols–Woronowicz algebra (or the symmetric algebra of the quantum space). Indeed, the composition law (12) on $\bigoplus_j \text{Im}(\mathcal{S}_j)$ can be written in the following equivalent form:

$$u \odot v := \mathcal{S}_{m+n}(u' \otimes v'), \tag{67}$$

where $u = \mathcal{S}_m u'$ and $v = \mathcal{S}_n v'$. Also, $\text{Im}(\mathcal{S}_j) \simeq V^{\otimes j}/\text{Ker}(\mathcal{S}_j)$, and the algebra on $\bigoplus_j \text{Im}(\mathcal{S}_j)$ can be defined alternatively as the algebra on $\bigoplus_j V^{\otimes j}/\text{Ker}(\mathcal{S}_j)$ with the composition law

$$\bar{u} \circ \bar{v} := u \otimes v \bmod \text{Ker}(\mathcal{S}_{m+n}), \tag{68}$$

where $\bar{u} \in V^{\otimes m}/\text{Ker}(\mathcal{S}_m)$ and $\bar{v} \in V^{\otimes n}/\text{Ker}(\mathcal{S}_n)$; $u \in V^{\otimes m}$ and $v \in V^{\otimes n}$ are representatives of $\bar{u}$ and $\bar{v}$, respectively. In formulations (67) or (68), the algebra on $\bigoplus_j \text{Im}(\mathcal{S}_j)$ or $\bigoplus_j V^{\otimes j}/\text{Ker}(\mathcal{S}_j)$ depends only on the collection $\{\mathcal{S}_j\}$; the composition laws (67) or (68) are well defined if $\mathcal{S}_{m+n}$ is divisible by $\mathcal{S}_m$ and $\mathcal{S}_n^{\uparrow m}$ from the right (which is, in general, weaker than $\mathcal{S}_{m+n} = \mathcal{T}_{n,m}\mathcal{S}_m\mathcal{S}_n^{\uparrow m}$); when, say, the representative u of $\bar{u}$ changes, $u \sim u + \delta u$, $\mathcal{S}_m(\delta u) = 0$, so $\delta u \otimes v \in \text{Ker}(\mathcal{S}_{m+n})$ and the product $\bar{u} \circ \bar{v}$ does not change, $u \otimes v \equiv (u + \delta u) \otimes v \bmod \text{Ker}(\mathcal{S}_{m+n})$.

However, the algebras on the space $\bigoplus_j V^{\otimes j}$, built with $\mathcal{T}'_{m,n}$ or $\mathcal{T}''_{m,n}$, are very different, as is seen, for example from the comparison of the spectra of the multiplicative and additive 1-shuffles in section 4. The collections $\{\mathcal{T}'_{m,n}\}$ and $\{\mathcal{T}''_{m,n}\}$ seem to have different ranges of applicability (already for the BMW algebras, the symmetrizers for these two collections do not coincide).

3.2. BMW algebras

The tower of the Birman–Murakami–Wenzl algebras $\text{BMW}_{M+1}(q, \nu)$ was introduced in [24] and [2]; it depends on two parameters, $q \in \mathfrak{k}^*$ and $\nu \in \mathfrak{k}\backslash\{0, q, -q^{-1}\}$. For $q^2 \neq 1$, the algebra $\text{BMW}_{M+1}(q, \nu)$ is the quotient of the braid group ring $\mathfrak{k}B_{M+1}$ by

$$\kappa_i \sigma_i = \sigma_i \kappa_i = \nu \kappa_i, \tag{69}$$

$$\kappa_i \sigma_{i-1} \kappa_i = \nu^{-1} \kappa_i, \quad \kappa_i \sigma_{i-1}^{-1} \kappa_i = \nu \kappa_i, \tag{70}$$

where elements κ_i are defined by

$$\sigma_i - \sigma_i^{-1} = (q - q^{-1})(1 - \kappa_i). \tag{71}$$

The Hecke quotient is $\kappa_i = 0$.

For $q^2 \neq 1$, the baxterized elements have the form ([20, 25, 16, 13])

$$\sigma_i(x) := x^{-1} \left(1 + \frac{x^2 - 1}{q - q^{-1}} \sigma_i + \frac{x^2 - 1}{1 - \nu^{-1} q^{-1} x^2} \kappa_i \right). \tag{72}$$

Their classical counterparts (for the Brauer algebras) were found in [35]. Elements (72) are normalized, $\sigma_i(1) = 1$, and satisfy the same unitarity conditions (50). The spectral decomposition of the generator σ_i contains three idempotents. The basic symmetrizer (the idempotent corresponding to the eigenvalue q) is proportional to $\sigma_i(q)$. However, $\sigma_i(q^{-1})$ is a mixture of two other idempotents. There are again isomorphisms $\iota : \mathrm{BMW}_{M+1}(q, \nu) \to \mathrm{BMW}_{M+1}(-q^{-1}, \nu), \sigma_i \mapsto \sigma_i$. Formula (72) is not invariant under ι. The basic anti-symmetrizer (the idempotent corresponding to the eigenvalue $-q^{-1}$) is proportional to $\iota^{-1}(\sigma(x))$ at $x = q$.

Again, the symmetrizers $S_n, n = 1, \ldots, M+1$, are the nonzero elements, which satisfy

$$S_1 = 1, \qquad S_n \in \mathrm{BMW}_n(q) \subset \mathrm{BMW}_{M+1}(q), \tag{73}$$

(52) and (53); they exist and are defined uniquely by conditions (73), (52) and (53).

Again, with the knowledge of the baxterized elements, one constructs the symmetrizers immediately: they are given by the same formula (54) and satisfy the same recurrence (55); the anti-symmetrizers are related to the symmetrizers by the isomorphisms ι.

The recurrency (56) holds for the BMW symmetrizers as well (it was used in [31, 7]); it is derived from the baxterized form of the symmetrizers by the same calculation (57).

Recurrency relation (58) does not hold for the BMW symmetrizers; the additive shuffles have to be modified. A version of such modification was suggested in [12] and can be derived by a calculation similar to (60). For the Hecke algebras the expansions of the products $\text{Ш}_{1,n-1}\text{Ш}_{1,n-2}\cdots\text{Ш}_{1,n-k}$ contain only reduced words; this is no longer so for the modified shuffles for the BMW algebras, the expansions contain similar terms (in a monomial basis, like the one suggested in [21]) and the formulae are not as elegant as for the Hecke algebras.

Formula (66) holds, with the same derivation, for the BMW symmetrizers.

4. Spectrum of 1-shuffles

Polynomial identities for the multiplicative (for the Hecke and BMW algebras) and additive (for the Hecke algebras) 1-shuffles follow, as a by-product, from (66) and (63). We establish the multiplicities of the eigenvalues in this section. The polynomial identity for the additive shuffle was discovered in [33] for the symmetric groups and generalized to the Hecke algebras in [22] with the help of the interpretation of the Hecke algebras in terms of flag manifolds over finite fields. The multiplicities of the eigenvalues of the additive shuffles were obtained in [6] for the symmetric groups. We propose a different approach to the calculation of the multiplicities for the Hecke algebras; our method uses the traces of the operators of the left multiplication by the additive shuffles.

Let $u \in H_n(q) \subset H_m(q), m \geqslant n$. Denote by L_u the operator of the left multiplication by u, $L_u : H_m(q) \to H_m(q), L_u(x) := ux$. We denote by $\mathrm{Tr}_{H_m}(L_u)$ the trace of the operator L_u, considered as the operator on $H_m(q)$.

(1) We start with the multiplicative shuffles. Since

$$^{q}\text{Ш}_{1,n} S_{n+1} = [n+1]_q! S_{n+1}, \tag{74}$$

we find, multiplying (66) by $(^{q}\text{Ш}_{1,n} - [n+1]_q!)$, the following polynomial identity for the multiplicative shuffle:

$$(^{q}\text{Ш}_{1,n})^n(^{q}\text{Ш}_{1,n} - [n+1]_q!) = 0, \tag{75}$$

which holds for both Hecke and BMW algebras. This is the minimal polynomial, already for the Hecke algebras. It is seen without calculations in the Burau representation [4] of H_{n+1},

$$\sigma_j(q^j) \mapsto [j+1]_q \,\mathrm{Id}_{j-1} \oplus \begin{pmatrix} q^{-j} & [j]_q \\ [j]_q & q^j \end{pmatrix} \oplus [j+1]_q \,\mathrm{Id}_{n-j}. \tag{76}$$

If the minimal polynomial is $t^i(t-[n+1]_q!)$ with $i<n$ (the eigenvalue $[n+1]_q!$ is present due to (74)) then S_{n+1} is proportional to the smaller than n power of ${}^q\text{Ш}_{1,n}$. The matrix of the element $\sigma_j(q^j)$ in the Burau representation has only one nonzero entry under the main diagonal, on the intersection of $(j+1)$st line and jth column. Therefore, the matrix of ${}^q\text{Ш}_{1,n}$ has only one sub-diagonal filled with (possibly) nonzeros. However, the matrix of S_{n+1} is proportional to the Hankel type matrix $A^i_j := q^{i+j}$; a smaller than n power of the matrix of ${}^q\text{Ш}_{1,n}$ has zero in the very left entry of the bottom line and cannot be equal to the matrix of S_{n+1}.

Thus, the element ${}^q\text{Ш}_{1,n}$ is not semi-simple for $n>1$; the semi-simple part of ${}^q\text{Ш}_{1,n}$ is $[n+1]_q!S_{n+1}$ and the eigenvalue $[n+1]_q!$ is simple (the rank of the projector $L_{S_{n+1}}$ on $H_{n+1}(q)$ is one, because $S_{n+1}\sigma_j = qS_{n+1}$, $j=1,\dots,n$).

(2) Similarly, multiplying (63) by $(\mathbf{Ш}_{1,n-1} - q^{1-n}[n]_q)$, we find the following polynomial identity for the additive shuffle:

$$(\mathbf{Ш}_{1,n-1} - q^{1-n}[n]_q)\prod_{j=0}^{n-2}(\mathbf{Ш}_{1,n-1} - q^{1-j}[j]_q) = 0. \tag{77}$$

The q-numbers $q^{1-j}[j]_q \equiv 1+q^2+\cdots+q^{2j-2}$, $j=1,2,\dots,$ are polynomials in q, linearly independent over $\mathbb{Z}$. Therefore, there is a unique integer combination $\sum_{j\in\{1,2,\dots,n-2,n\}} n_j q^{1-j}[j]_q$, $n_j\in\mathbb{Z}$, of these q-numbers, which is equal to the trace of $L_{\text{Ш}_{1,n-1}}$; the coefficients n_j in this combination are the multiplicities of the eigenvalues $q^{1-j}[j]_q$, $j>0$. The multiplicity n_0 of the eigenvalue 0 is fixed by $\sum n_j = \dim(H_n(q)) \equiv n!$. Thus, the presence of the parameter q gives a simple way to calculate the multiplicities.

Lemma 1. *(i) If $u\in H_j$ then*

$$\mathrm{Tr}_{H_{j+1}}(L_u) = \mathrm{Tr}_{H_{j+1}}(L_{u^{\uparrow 1}}) = (j+1)\mathrm{Tr}_{H_j}(L_u). \tag{78}$$

$$\text{(ii)}\quad \mathrm{Tr}_{H_{j+1}}(L_{\sigma_1\sigma_2\cdots\sigma_j}) = (q-q^{-1})\mathrm{Tr}_{H_j}(L_{\sigma_1\sigma_2\cdots\sigma_{j-1}}), \qquad j>0. \tag{79}$$

$$\text{(iii)}\quad \mathrm{Tr}_{H_j}(L_{\sigma_{k-l+1}\cdots\sigma_{k-1}\sigma_k}) = \frac{j!}{(l+1)!}(q-q^{-1})^l, \qquad j>k\geqslant l. \tag{80}$$

Proof. Recall that, as a vector space, $H_{j+1}(q)\simeq \oplus_{k=-1}^{j-1}W_k$, where W_k is the vector space consisting of elements $v\sigma_j\sigma_{j-1}\cdots\sigma_{j-k}$ with $v\in H_j(q)$ (the word $\sigma_j\sigma_{j-1}\cdots\sigma_{j-k}$ is, by definition, empty for $k=-1$); each W_k is canonically isomorphic to $H_j(q)$ as a vector space, the isomorphism is $v\sigma_j\sigma_{j-1}\cdots\sigma_{j-k}\mapsto v$. The Hecke versions of the automorphism $\mathfrak{a}$ and the anti-automorphism $\mathfrak{b}$, defined in (6), transform the above decomposition of $H_{j+1}(q)$ into $H_{j+1}(q)\simeq\oplus_{k=-1}^{j-1}W'_k$, where W'_k consists of elements $v\sigma_1\sigma_2\cdots\sigma_{k+1}$ with $v\in H_j(q)^{\uparrow 1}$ and $H_{j+1}(q)\simeq\oplus_{k=-1}^{j-1}W''_k$, where W''_k consists of elements $\sigma_{j-k}\cdots\sigma_{j-1}\sigma_j v$ with $v\in H_j(q)$.

The operator L_u (respectively, $L_{u^{\uparrow 1}}$) acts in each of the spaces W_k (respectively, W'_k) separately and this action commutes with the isomorphisms $W_k\simeq H_j(q)$ (respectively, $W'_k\simeq H_j(q)^{\uparrow 1}$). This establishes (i).

In fact, it was enough to find the formula for $\mathrm{Tr}_{H_{j+1}}(L_u)$; $\mathrm{Tr}_{H_{j+1}}(L_u) = \mathrm{Tr}_{H_{j+1}}(L_{u^{\uparrow 1}})$ because $u^{\uparrow 1}$ is conjugate to u, $\sigma_1 \cdots \sigma_j u = u^{\uparrow 1}\sigma_1 \cdots \sigma_j$, for $u \in H_j(q)$.

(ii) Given a basis $\{e_\alpha\}$ of a vector space U we say that the vector e_α (from the basis) does not contribute to the trace of an operator $X : U \to U$ if $X^\alpha_\alpha = 0$ (no summation), where X^β_α is the matrix of X in the basis $\{e_\alpha\}$.

We use the decomposition $H_{j+1}(q) \simeq \oplus_{k=-1}^{n-1} W''_k$. The operator $L_{\sigma_1\sigma_2\cdots\sigma_j}$ maps W''_{-1} to W''_{j-1}, so vectors from W''_{-1} do not contribute to the trace of $L_{\sigma_1\sigma_2\cdots\sigma_j}$. For $0 \leqslant k < j$,

$$\begin{aligned}
(\sigma_1 \cdots \sigma_j)(\sigma_{j-k} \cdots \sigma_{j-1} \cdot \sigma_j)v &= (\sigma_{j-k+1} \cdots \sigma_j)(\sigma_1 \cdots \sigma_j)\sigma_j v \\
&= (\sigma_{j-k+1} \cdots \sigma_j)(\sigma_1 \cdots \sigma_{j-1})((q - q^{-1})\sigma_j + 1)v \\
&= (q - q^{-1})(\sigma_{j-k+1} \cdots \sigma_j)(\sigma_1 \cdots \sigma_j)v + (\sigma_{j-k+1} \cdots \sigma_j)(\sigma_1 \cdots \sigma_{j-1})v \\
&= (q - q^{-1})(\sigma_1 \cdots \sigma_j)(\sigma_{j-k} \cdots \sigma_{j-1})v + (\sigma_{j-k+1} \cdots \sigma_j)(\sigma_1 \cdots \sigma_{j-1})v,
\end{aligned} \tag{81}$$

the operator $L_{\sigma_1\sigma_2\cdots\sigma_j}$ maps W''_k to $W''_{j-1} \oplus W''_{k-1}$. Therefore, vectors from W''_k do not contribute to the trace of $L_{\sigma_1\sigma_2\cdots\sigma_j}$ for $k < j - 1$. For $k = j - 1$, the component $L^\diamond_{\sigma_1\sigma_2\cdots\sigma_j}$ of the operator $L_{\sigma_1\sigma_2\cdots\sigma_j}$, which maps W''_{j-1} to W''_{j-1}, may have a nonzero trace. This component reads, by (81),

$$L^\diamond_{\sigma_1\sigma_2\cdots\sigma_j}(\sigma_1 \cdots \sigma_j v) = (q - q^{-1})\sigma_1 \cdots \sigma_j L_{\sigma_1\sigma_2\cdots\sigma_{j-1}}(v),$$

and the assertion (ii) follows.

(iii) Follows from (i) and (ii). □

By (80), the trace of the operator $L_{\coprod_{1,n-1}}$ is

$$\mathrm{Tr}_{H_n}(L_{\coprod_{1,n-1}}) = \sum_{i=0}^{n-1} \frac{n!}{(i+1)!}(q^2 - 1)^i. \tag{82}$$

For the symmetric group $\mathbb{S}_n$, the multiplicity of the eigenvalue j of $L_{\coprod_{1,n-1}}$ is the number of permutations in $\mathbb{S}_n$ with exactly j fixed points [6]. Recall that the derangement number d_n (the number of permutations in $\mathbb{S}_n$ without fixed points) is

$$d_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \tag{83}$$

and the number $d_{n,j}$ of permutations in $\mathbb{S}_n$ with exactly j fixed points is

$$d_{n,j} = \binom{n}{j} d_{n-j} \equiv \frac{n!}{j!} \sum_{i=0}^{n-j} \frac{(-1)^i}{i!}. \tag{84}$$

For generic q, the multiplicities of the eigenvalues of $L_{\coprod_{1,n-1}}$ are the same as for the symmetric group. By construction, $\sum_{j=0}^{n} d_{n,j} = n!$. Thus, to rederive the multiplicities we have only to check that $\sum_{j=0}^{n} d_{n,j} q^{1-j}[j]_q = \mathrm{Tr}_{H_n}\big(L_{\coprod_{1,n-1}}\big)$, or, explicitly,

$$\sum_{j=0}^{n} \frac{n!}{j!} \sum_{k=0}^{n-j} \frac{(-1)^k}{k!} q^{1-j}[j]_q = \sum_{i=0}^{n-1} \frac{n!}{(i+1)!}(q^2 - 1)^i. \tag{85}$$

It is straightforward to verify that both left- and right-hand sides satisfy the same recurrency relation in n

$$f_{n+1} = (n+1) f_n + (q^2 - 1)^n, \tag{86}$$

with the same initial condition $f_0 = 0$, and thereby coincide.

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